An Extension of the Note-Based Tonnetz to 31-Tone Equal Temperament

First Author
Affiliation
author@ul.ie

ABSTRACT

When musicians approach composing in the visual domain, they are usually working with the traditional notation of markings on staff paper. While other visual representations do exist, like those that leverage the geometric relationships of notes, these models are usually created, and used in the context of musical analysis. However, the visual properties of these geometric representations can be also utilized in the process of musical creation, which is particularly useful when composing in unfamiliar conditions, like microtonal tuning systems. In this paper, I propose the use of the Tonnetz as a compositional tool in 12-tone ET, and its extension to 31-tone ET.

1. BACKGROUND

In *A Geometry of Music*, Dimitri Tymoczko defines a series of chord-based graphs. These basic versions represent a geometric model of n-dimensional voice leading, where the n refers both to the spatial dimension of the structure, as well as the amount of notes in each chord. In both of these examples, the notes that make up the chords are undordered, and agnostic of register, and thus they can be wrapped into a repeating structure, like a mobius strip in the 2-dimensional case.

While the movement of individual notes can be inferred from these chords, it is difficult to conceptualize. To model the intervalic relationships of the individual notes themselves, we can create a similar structure, where a node represents a single note, and an edge represents the interval between two notes. In its generalized form, this is called a *Tone Network*, or in the more common German translation, a *Tonnetz*.

2. THE TONNETZ

2.1 Basic Structure

The basic foundations of the *Tonnetz* rely on 3 intervals, the major third, the minor third, and the perfect fifth[1]. As seen in figure 1, moving to the right in the horizontal direction ascends by a *perfect fifth*, moving up at a 60angle ascends by a *major third*, and moving downwards, at a 300angle, descends by a *minor third*. It is clear from this diagram that these three notes form a tradic chord, specifically a C major triad. The utility of this structure is that

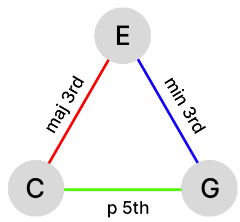

Figure 1. The basic structural building block

it allows us to model *Neo-Riemannian* transformations, by inverting these notes over one of its 3 edges.

Neo-Riemannian Theory refers to a set of writings and ideas proposed by Hugo Riemann, and extended by modern theorists like David Lewin, and Richard Cohn [2]. The work is based on a series of basic transformations that describe the overlap between triads, regardless of their diatonic relationship. These transformations, *Parallel*, *Leading-Tone*, and *Relative*, describe two chords that share a perfect fifth, minor third, and a major third, respectively [2]. Since these shared intervals, all represent an edge on our triangle, we can map this relationship onto our structure. In the case of the *Parallel* transformation, we can add the note Eb to our structure, such that there is an edge connecting C-Eb that is parallel to the E-G edge, and that there is an edge connecting G-Eb that is parallel to the C-E edge. Figure 2, shows that the resulting structure, looks like the mirror image of our original triangle. In fact, this intuition reveals another useful property, that each of these transformations over an edge, can also be modeled as an OPTIC inversion, as defined by Tymoczko [3]. Specifically, the axis of inversion is the axis that evenly subdivides the two notes in the transformation's shared edge. In this case, this axis is I_E^{Eb}, which cuts the perfect fifth interval of C-G in half.

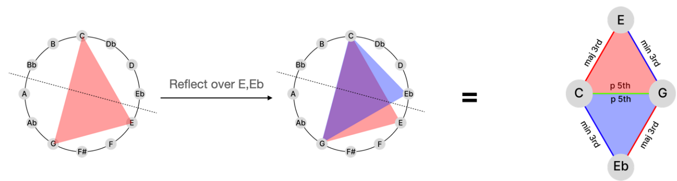

Figure 2. A Neo-Riemannian Parallel transformation

If we recursively repeat these transformations, for each edge of the triangle, we can create the full, infinite *Tonnetz*, that models each of the 3 transformations, for every possible basic triad. Because there only 24 possible basic triads in 12-ET (major/minor for each of the 12 tones), this structure tends to repeat relatively quickly. If we start at

Copyright: ©2024 First Author et al. This is an open-access article distributed under the terms of the Creative Commons Attribution License 3.0 Unported, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.

C major, it will only require 8 transformations leftward, to arrive back at C major. Because of this infinitely repeating property, The *Tonnetz* can be folded, so that the bolded lines touch, creating a toroidal structure [1].

This note-based structure can even be abstracted another level, to represent the chords themselves. This is referred to as the "chicken-wire" *Tonnetz*, initially proposed by Jack Douthett and Peter Steinbach [4]. To create the chord-based abstraction, a new node can be placed in the center of each triangle, and then connected to the other 3 chords in which it shares a transformation with. This process is visualized in Figure 3. Now, the nodes represent chords, and the edges represent the transformations. Like its note-based cousin, this structure is also repeatable, and can again be folded over the Torus.

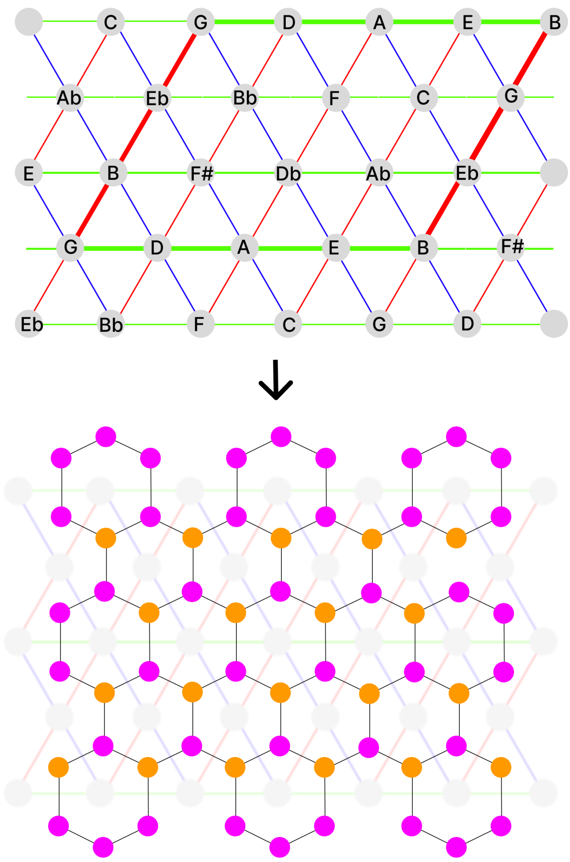

Figure 3. Construction of the chord-based dual graph

2.2 Compositional Use

As the *Tonnetz* provides a visual framework for the harmonic overlap between chords, it can be used as a compositional roadmap, that can help a composer create harmonic support for a melody, transition to a target chord, or simply break natural composition tendencies that they may have when compositing strictly in notation. For example, if a composer wanted to migrate tonal centers from Cmajor to Bmajor, they might be tempted to simply transpose down by a semitone. However, by using the *Tonnetz* as a framework, they can reach the same destination as transposition, instead by chaining together inversions, perhaps leading to a more harmonically rich, and interesting transition.

3. EXTENSION TO 31-TONE EQUAL TEMPERAMENT

The visually-based compositional capabilities of this structure lend itself well to unfamiliar conditions, which for myself, and likely many western composers, describes microtonal composition well. Because the *Tonnetz* structure requires consistent intervals in a given direction, it would not apply well to any non-equal temperament scale, as there would be significant pitch drift every time we completed one repetition of single Tonnetz tile. However, this makes the Tonnetz a perfect candidate for any equal temperament scale, regardless of how many notes it has. In this case, I chose to study the 31-note ET scale as there is not much current literature that describes its relationship to these geometric tools.

The 31-tone ET system is composed of single octave that is equally divided by 31 tones. In this tuning, the frequency ratio between consecutive tones is $2^{\frac{1}{31}}$, or 38.71 cents[5].

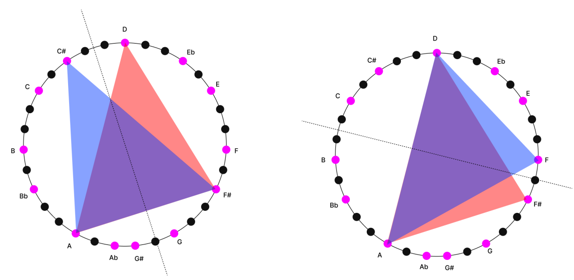

Figure 4. Inversions in 31-ET space

Since we have consistent intervals, the inversions outlined above in section 3.1, also hold true for 31-ET, as seen in 4. This means that the basic *Neo-Riemannian* transformations will also hold true. Thus, the Tonnetz can be generated using the same interval structure as 12-tone, except now a perfect fifth is 18 steps, a minor third is 8 steps, and a major third is 10 steps. The only difference from 12-tone, as seen in Figure 5 is that our repeating tile will be much bigger, since now there are many more possible chords - 62 to be exact.

While the size of the tile is not in itself a limiting factor, the types of chord qualities that can be modeled is somewhat limiting to our use-case. In 12-Tone, there are only 2 basic triads, so our Neo-Riemmannian Tonnetz is fairly comprehensive of many possible harmonic sequences. However, in 31-tone, there are 3 additional basic triad qualities. These triads are downminor (tone steps 0-7-18), mid (0-9-18), and upmajor (0-11-18)[5]. Thus, in order to fully leverage the benefits of 31-tone harmony, the Tonnetz structure needs the ability to frequently lower, or raise a note by a single tone. In our current basic Tonnetz structure, this is not possible, as this is not accounted for by the traditional 12-tone Neo-Riemannian theory. Any of these three triads, cannot be created, without breaking the structure of our Tonnetz. To solve this, I propose the modification of this

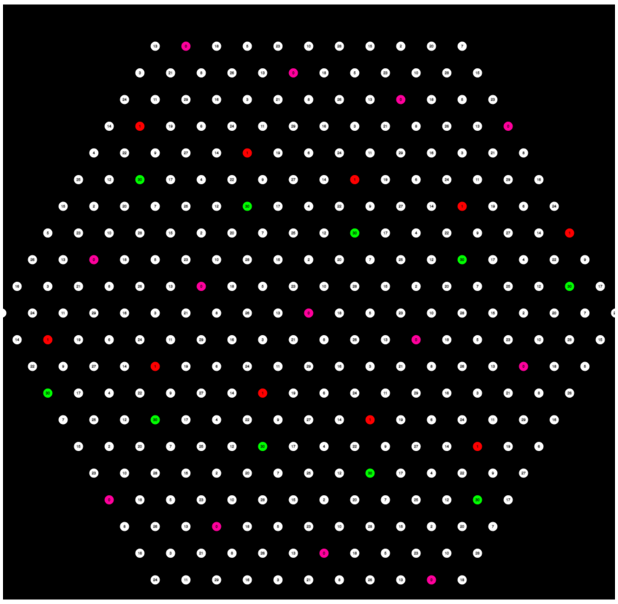

Figure 5. The 31-ET Tonnetz

2d structure, to include a z (depth) dimension. At each step in the z direction, A new Tonnetz would be placed, where every note in the network is transposed by a single step, such that shifting a single node backwards in the Z direction would transpose it down by a step, and shifting it forwards in the Z direction would transpose it up by a step. A slice of this structure is shown in Figure 6, where the notation ”vC” means ”down C”, representing a downward step wise transposition. The bolded lines represent the Tonnetz edges, while the thin lines are just spatial markers to model depth. It is implied that an additional layer, instead starting from ”⋀C” or ”up C”, would be placed *in front* of the original, orange layer, in the opposite direction. These layers would extend, like the *Tonnetz*, into infinity, requiring 31 steps in z direction, to return the original. Figure 7 models a slice of this multi layer stacking.

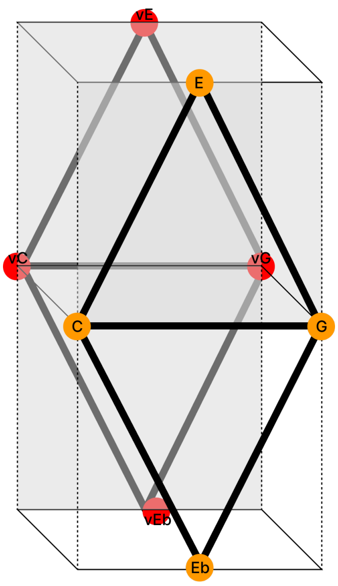

Figure 6. A single chordal slice of the proposed 31-Tone Tonnetz

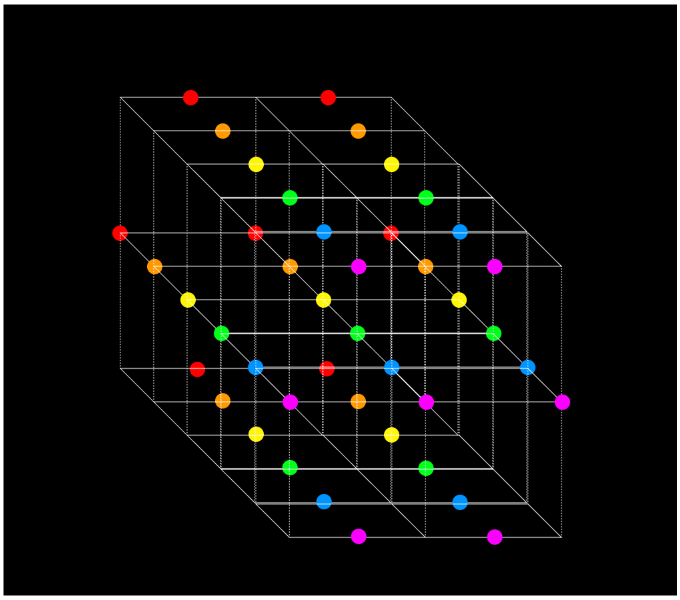

Figure 7. Slice of the multi-layer Tonnetz

With this structure, it is now possible to efficiently, visually create harmonic, inversion-centric sequences in the Tonnetz structure, whilst also leveraging all five of the basic triads in 31-tone equal temperament. Shown in Figure 8 is a basic example of a chord progression, created with this new extended structure. From right to left, the chords are C major, C mid (or C neutral), Down C major, and Down C Up Major (or Super Major).

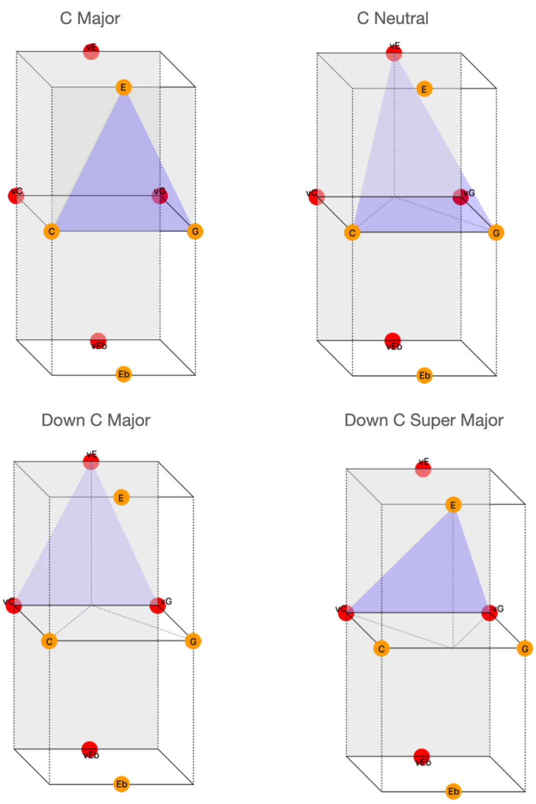

Figure 8. An example chord progression in 31-ET

4. CONCLUSION AND FUTURE WORK

While this new Tonnetz is useful, there are a few limitations. Mainly, it has not yet become clear that there is a way to tile this structure, like the torus, without a need for a fourth dimension. Currently, in my definition of the structure, an infinite plane, extending in all directions must be assumed to conceptualize its use. However, in the context of a virtual 3D software space, like a VR/AR environment, this is not much of an issue. In fact, this context could bolster the compositional benefits that this tool provides, and is likely where a model like this will thrive the most.

5. REFERENCES

[1] D. Tymoczko, “The Generalized Tonnetz,” p. 1–52, Apr. 2012. [Online]. Available: http://dx.doi.org/10.1215/00222909-1546958

[2] R. Cohn, “Neo-Riemannian Operations, Parsimonious Trichords, and Their ”Tonnetz” Representations,” *Journal of Music Theory*, vol. 41, no. 1, pp. 1–66, 1997. [Online]. Available: http://www.jstor.org/stable/843761

[3] D. Tymoczko, *Geometry of Music Harmony and counterpoint in the extended common practice*. Oxford University Press, USA, 2014.

[4] J. Douthett and P. Steinbach, “Parsimonious Graphs: A Study in Parsimony, Contextual Transformations, and Modes of Limited Transposition,” *Journal of Music Theory*, vol. 42, no. 2, pp. 241–263, 1998. [Online]. Available: http://www.jstor.org/stable/843877

[5] “31edo - Xenharmonic Wiki.” [Online]. Available: https://en.xen.wiki/w/31edo