

Fractals in Music

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Fractals are regarded as one of many core intersections between mathematics and nature. They are an example of mathematics as a sense and way of seeing the world, beyond mere equations and graphs, from sea shells to networks of rivers. What is also clear are the intersections between music and nature, from the natural sounds of indigenous music, to environmental soundscapes. Naturally, fractals' application to music has become an evermore interesting topic regarding their musical and mathematical aspects, as well as their practical use in today's world.

Any pattern that is endless and self-similar at different scales following a recursive structure is considered a fractal. The concept of recursion is one that is core to computer science, where a function can call itself repeatedly until a base case is reached. A basic example is a tree, where the trunk splits into branches, and those branches can split up to smaller branches in a similar fashion to the trunk split. Such patterns could continue over many iterations.

```
def factorial(n):  
    if n == 0:  
        return 1  
    else:  
        return n * factorial(n-1)
```

Fig. 1. Factorial, a mathematical operation denoted by  $n!$  |  $n \in \mathbb{N}$  that uses recursion by calling a smaller instance of itself, e.g.  $5! = 5 \cdot 4!$ ,  $4! = 4 \cdot 3!$ , ...,  $0! = 1$ , as a function in Python.

One could utilize a similar pattern in a musical composition in many ways. One way is to fractalize the structure of the music such that the speed of a melody becomes so fast that it has a

pitch that could itself be used to make the same melody.<sup>1</sup> This can be done mathematically utilizing the idea that notes can be built using ratios from a root, i.e. just intonation. Given a just intonation scale, one could assign each note a natural number  $N$  that is consistent with each of their ratios. One could then fit  $N$  occurrences of the melody within a note depending on its duration, maintaining consistency, e.g.  $N$  occurrences within an eighth note,  $2N$  occurrences within a quarter note (as the note duration is twice as long), etc. Those notes should match the notes in the original melody. What results is the melody with “itself in itself.” It can be heard at the original tempo and a slower tempo.<sup>2</sup> This process could be theoretically done forever like in an ideal, infinite fractal, however there may be technical restrictions. In Ableton Live, which one could use to do the aforementioned process, the CPU load is displayed, and after trying to do a second iteration, the pitch sounds glitched (in most cases) as the CPU goes way over 100%, a reasonable and expected consequence. To avoid this, one could instead export notes for the first iteration to audio files and use them to create notes for the second iteration, thereby making this fractal process repeat truly infinitely.

Note Name	Ratio	Number $N$
D (Root)	$\frac{1}{1}$	60
F	$\frac{6}{5}$	72
F#	$\frac{5}{4}$	75
G	$\frac{4}{3}$	80

<sup>1</sup> Charly Sauret, *Fractals in Music - Exploration & Experiment*, YouTube, 2022, <https://youtu.be/6XTe5EsPE20>, 7:10.

<sup>2</sup> Adam Neely, *Music Fractals*, YouTube, 2017, <https://youtu.be/mq0z-sxjNlo>, 0:47.

G#	$\frac{7}{5}$	84
A	$\frac{3}{2}$	90
A#	$\frac{8}{5}$	96
B	$\frac{5}{3}$	100
D	$\frac{2}{1}$	120

Table 1. For a composition containing D, F, F#, G, G#, A, A#, & B as pitches, one could utilize their corresponding ratios in just intonation based on D as the root note and assign them a natural number such that their proportions are maintained.<sup>3</sup> For example, The ratio between

A# and D is  $\frac{8}{5}$ , which is maintained with  $\frac{N_{A\#}}{N_D} = \frac{96}{60} = \frac{8}{5}$ .  $N_D$  can easily be calculated as the

least common multiple (LCM) of all the denominators:  $LCM(1, 5, 4, 3, 2) = 60$ .  $N$  is sometimes referred to as a harmonic number of some pitch; further explanation of this is avoided as it is slightly tangential to the discussion of fractals.

There are many other ways to fractalize a composition. Fractalizing a melody itself entails a track having a repeating melody and other tracks having that same melody stretched or shrunk twice in duration.<sup>4</sup> This was the technique done in the creative project instead of the one aforementioned, since it utilized a 13-limit just intonation scale with an LCM of 360360, therefore causing difficulty in fractalizing its structure. Fractalizing the timbre is done by applying the same change to each iteration of a melody, perhaps any effects done on it (e.g. reverb) or its pitch/octave.<sup>5</sup> Fractalizing the speed utilizes a similar process as a shepard's tone.

<sup>3</sup> Neely, 3:15.

<sup>4</sup> Sauret, 1:53.

<sup>5</sup> Sauret, 3:34.

One could have three tracks with the same melody, one at twice the speed, one at normal speed, and one at half the speed. The one at half the speed could slowly fade in and reach the speed of the one at normal speed, the one at normal speed could slowly reach the speed of the one at twice the speed, and the one at twice the speed could slowly fade out, producing an illusion where the piece seems to be increasing in tempo infinitely.<sup>6</sup>

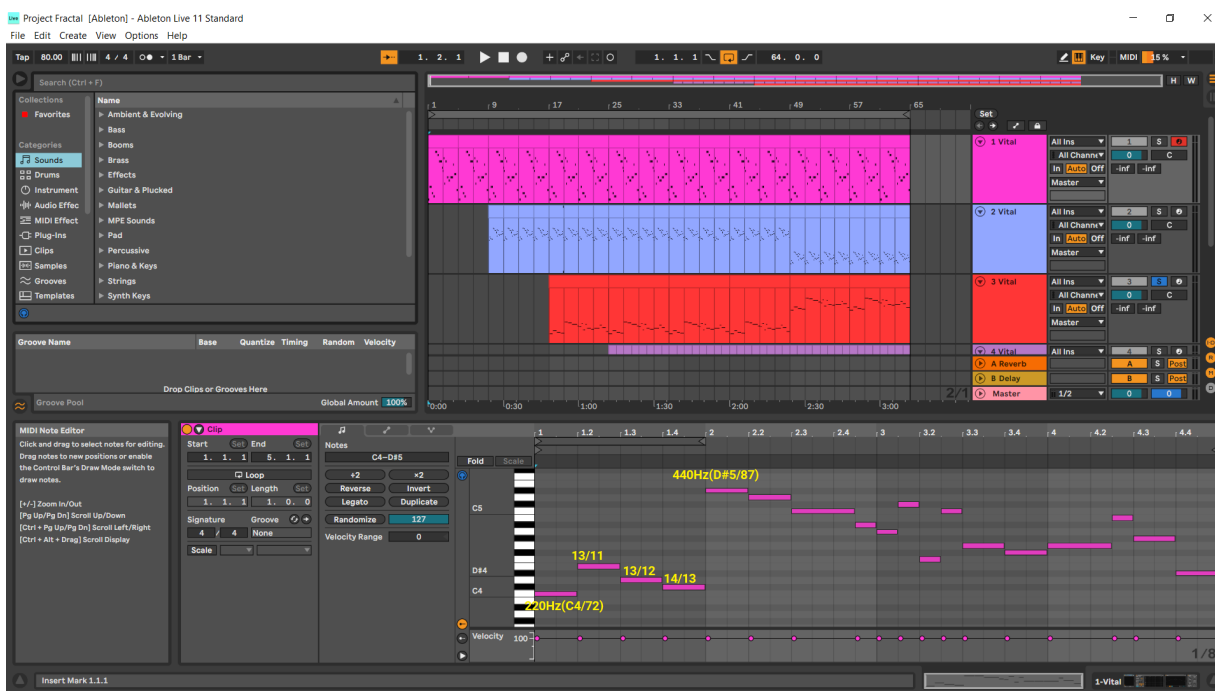


Fig. 2. A brief overview of the creative project. Illustrated is the fractalization of the melody itself, containing versions of itself stretched  $2^n \mid n \in \{-2, -1, 0, 1, 2\}$  times. Labeled are the ratios of each note to the root. Note their denominators due to using 13-limit 15-note JI.

Fractals and its presence in music have continued to be studied to this day. One concept in particular that is of interest is fractal Brownian motion (fBm). Regular Brownian motion is simply the motion or fluctuation of a particle in a medium by incrementing or decrementing by

<sup>6</sup> Sauret, 4:55.

some amount.<sup>7</sup> Compared to regular Brownian motion, where such increments are independent, fBm has increments that are dependent within time (regular steps).<sup>8</sup> This is represented by the Hurst exponent  $H$ , where  $H = \frac{1}{2}$  represents regular Brownian motion,  $H > \frac{1}{2}$  represents positive correlation within increments of fBm, and  $H < \frac{1}{2}$  represents negative correlation. What had been repeatedly noted was the relationship between fractional Brownian motion (fBm) and music. Through the Hurst exponent and Fourier spectral analyses, it was revealed that musical melodies predominantly exhibit  $1/f$  noise, which describes a spectral density inversely proportional to frequency.<sup>9</sup> The power spectrum of music sequences reflect a  $\frac{1}{f^\beta}$ -type spectrum, indicating a fractal nature in the tone changes over different temporal scales. This is because they tend to display self-similar properties at different scales within their time-frequency analysis.<sup>10</sup>




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<sup>7</sup> Matthew Ward, "Fractional Brownian Motion (fBm)," Worcester Polytechnic Institute, February 20, 1999, <https://davis.wpi.edu/~matt/courses/fractals/brownian.html>.

<sup>8</sup> Eric Rauch, "Fractal Brownian Archipelago," Massachusetts Institute of Technology, 2003, <https://groups.csail.mit.edu/mac/users/rauch/islands/>.

<sup>9</sup> Richard F. Voss, "Random Fractals: Self-Affinity in Noise, Music, Mountains, and Clouds," *Physica D: Nonlinear Phenomena* 38, no. 1–3 (September 1989): 362–71, [https://doi.org/10.1016/0167-2789\(89\)90220-0](https://doi.org/10.1016/0167-2789(89)90220-0).

<sup>10</sup> Zhi-Yuan Su and Tzuyin Wu, "Music Walk, Fractal Geometry in Music," *Physica A: Statistical Mechanics and Its Applications* 380 (July 1, 2007): 418–28, <https://doi.org/10.1016/j.physa.2007.02.079>.



Fig. 3. A real-life example of fBm. The first image displays fBm with  $H = 0.7$ . The second image displays a Fractal Brownian island meant to be similar to the coast of Great Britain, which clearly follows a similar shape to fBm.<sup>11</sup>

The fractal dimension  $D$ , determines how complicated a fractal is.  $N = S^D$  and  $D = \frac{\log(N)}{\log(S)}$ , where  $N$  is the number of self similar pieces and  $S$  is the scale factor which would be used to produce the same fractal again (i.e. the ratio between the original set and a subset).<sup>12</sup> A second intriguing fractal music concept is the application of all this terminology on music to find a fractal relation. This came from the relation of an earthquake, which is  $F = \frac{c}{M^D}$ , where  $F$  is the frequency,  $M$  is a parameter representing intensity, and  $c$  is some constant of proportionality, and  $D$  is the fractal dimension.<sup>13</sup> Successions of notes in music would be considered fractal with the

<sup>11</sup> Rauch.

<sup>12</sup> Alexey Nogin, "Mathematical Interpretation of Fractal Dimension," Cornell University, 1998, <https://www.cs.cornell.edu/courses/cs212/1998sp/handouts/Fractals/similar.html>.

<sup>13</sup> Kenneth J. Hsü and Andreas J. Hsü, "Fractal Geometry of Music.," Proceedings of the National Academy of Sciences 87, no. 3 (February 1, 1990): 938–41, <https://doi.org/10.1073/pnas.87.3.938>, 938.

equation  $F = \frac{c}{i^D}$ , where  $F$  is the incidence frequency of note interval  $i$  and  $i$  is the interval

between successive notes in a composition, e.g.  $i = 1$  in a semitone,  $i = 2$  in a full tone, etc.

There is bound to be some deviations from this equation from real music, and one can model an approximate relationship to be linear on a *log-log* graph, where the horizontal axis is  $i$  and the vertical axis is  $\log(F)$ . This provides a unique fractal-relation visualization of music.<sup>14</sup>

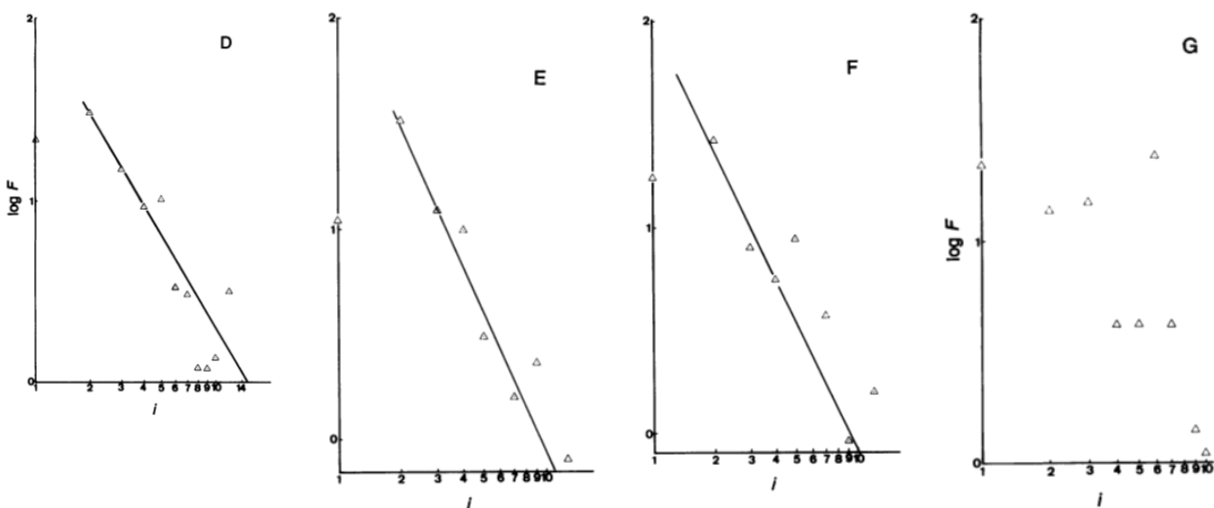


Fig. 4. *log-log* graphs demonstrating fractal relationships in different pieces. The are of:

- D - Mozart KV 533, with fractal relation  $F = \frac{1}{i^{1.7322}}$ ,
- E - Six Swiss children's songs, with notable note repetition,
- F - Mozart KV 331, with similar note repetition and therefore a similar fractal relation to E,
- G - Stockhausen's "Capricorn," a more modern piece, with no fractal relation apparent.<sup>15</sup>

<sup>14</sup> Hsü, 939.

<sup>15</sup> Hsü, 940.

Finally, researchers have wanted to focus on fractal patterns in melodic lines that recur over various temporal scales, i.e. different time scales. This is adjacent to the aforementioned compositional technique that was coined simply “fractalizing the melody,” though in this case temporal scaling can include scaling beyond simple doubling or halving. The temporal dimension  $D_0$  is generally defined similarly to before,  $D_0 = \frac{\log(N)}{\log(S)}$ , where  $N$  is the number of notes of a motif which is replaced by  $N$  notes of  $\frac{1}{S}$  duration. This can be combined with the tonal dimension  $D_t$ , which is used to describe its melodic shape to account for pitch/tonal variations, to obtain the overall fractal dimension of  $D = D_0 + D_t$ .<sup>16</sup> These concepts can assist in determining the sophistication and intricacies of self-similar patterns within a composition’s organization.



Fig. 5. The opening measures of Josquin des Prez’s *Missa L’homme armé super voces musicales*, Agnus Dei II. The ratios of the speeds of the 3 voices, soprano, alto, and tenor are 3:1:2, respectively.  $D = D_0 + D_t = 2 + 0.64 = 2.64$  via equations not discussed in this paper.<sup>17</sup> Prolation canons by Josquin, Bach, and Brahms mostly have  $2 \leq D \leq 4$ . The first

<sup>16</sup> John McDonough and Andrzej Herczyński, “Fractal Patterns in Music,” *Chaos, Solitons & Fractals* 170 (2023): 1–13, <https://doi.org/10.1016/j.chaos.2023.113315>, 11.

<sup>17</sup> McDonough, 7–8.

line is transcribed to have a different time signature to emphasize the speed ratios.

This paper serves as an overview for readers to utilize fractals in music, both compositionally and analytically. It aims to emphasize mathematics and music's interconnection, seemingly by way of nature. It is a brief but clear examination of research in this field throughout the years and more recently. The idea of self-similarity could be applied to essentially any parameter of a music composition, illustrating the possibilities of this unique idea, although it is important to note its limitations. Analyses of music's fractal nature have entailed the use of fBm, adjacent fractal equations, and different perspectives of the fractal dimension. Composers should view their work through the lens of both natural and intentional fractal properties in music. This field ultimately provides an opportunity for exploration of the inner workings of mathematics and music, as well as a way to push the boundaries of what is conventional and effective in a musical composition.

## Bibliography

- Hsü, Kenneth J., and Andreas J. Hsü. “Fractal Geometry of Music.” *Proceedings of the National Academy of Sciences* 87, no. 3 (February 1, 1990): 938–41.  
<https://doi.org/10.1073/pnas.87.3.938>.
- McDonough, John, and Andrzej Herczyński. “Fractal Patterns in Music.” *Chaos, Solitons & Fractals* 170 (2023): 1–13. <https://doi.org/10.1016/j.chaos.2023.113315>.
- Nogin, Alexey. “Mathematical Interpretation of Fractal Dimension.” Cornell University, 1998.  
<https://www.cs.cornell.edu/courses/cs212/1998sp/handouts/Fractals/similar.html>.
- Neely, Adam. *Music Fractals*. YouTube, 2017. <https://youtu.be/mq0z-sxjNlo>.
- Rauch, Eric. “Fractal Brownian Archipelago.” Massachusetts Institute of Technology, 2003.  
<https://groups.csail.mit.edu/mac/users/rauch/islands/>.
- Sauret, Charly. *Fractals in Music - Exploration & Experiment*. YouTube, 2022.  
<https://youtu.be/6XTe5EsPE20>.
- Su, Zhi-Yuan, and Tzuyin Wu. “Music Walk, Fractal Geometry in Music.” *Physica A: Statistical Mechanics and its Applications* 380 (July 1, 2007): 418–28.  
<https://doi.org/10.1016/j.physa.2007.02.079>.
- Voss, Richard F. “Random Fractals: Self-Affinity in Noise, Music, Mountains, and Clouds.” *Physica D: Nonlinear Phenomena* 38, no. 1–3 (September 1989): 362–71.  
[https://doi.org/10.1016/0167-2789\(89\)90220-0](https://doi.org/10.1016/0167-2789(89)90220-0).
- Ward, Matthew. “Fractional Brownian Motion (fBm).” Worcester Polytechnic Institute, February 20, 1999. <https://davis.wpi.edu/~matt/courses/fractals/brownian.html>.